A 1D bar problem

Task:

* Solve this problem with different number of elements! *

Background and analytical solution

Consider an linear elastic bar of length $L$ with a varying cross section $A(x)$

$$ A(x) = A_0 \left(1 - \frac{x}{2L}\right)^2 $$

and a given Young’s modulus $E$. The bar is rigidly supported in the left end and in the right acts a concentrated force $F$. The cross section variation is defined as follows

![Figure 1: The given bar geometry](image)

The exact solution for the displacement $u(x)$ can be achieved by solving of the strong formulation defined in box $\mathcal{S}$ where the given displacement $g$ and the distributed force/unit length $b(x)$ are equal to zero. Putting in the cross section expression into the differential equation gives

$$ \frac{d}{dx} \left( A_0 \left(1 - \frac{x}{2L}\right)^2 E \frac{du(x)}{dx} \right) = 0 $$

and one integration of both sides $\Rightarrow$

$$ EA_0 \left(1 - \frac{x}{2L}\right)^2 \frac{du(x)}{dx} = C_1 $$

where $C_1$ is a unknown constant. In the right end we have the boundary condition

$$ EA(L) \frac{d u(L)}{dx} = A(L) h = F \quad \Rightarrow \quad C_1 = F. $$

A second integration then gives

$$ u(x) = \frac{2FL}{EA_0 \left(1 - \frac{x}{2L}\right)} + C_2 $$
where the boundary condition \( u(0) = 0 \) \( \Rightarrow \)
\[
C_2 = -\frac{2FL}{EA_0}
\]
and the analytical solution for the displacement \( u(x) \) is
\[
u(x) = \frac{x}{2L} \frac{2FL}{(1 - \frac{x}{2L}) EA_0}
\]
and for the stress \( \sigma(x) \) we have
\[
\sigma(x) = E \frac{du(x)}{dx} = \frac{1}{(1 - \frac{x}{2L})^2 A_0} F.
\]
This expression will be used later on as a comparison.

Let us now solve this problem numerically by making use of the finite element method. Use the following finite element mesh consisting of three 2-node 1D elements of equal length \( L^e_i = L/3 \) where \( i = 1, 2, 3 \). Concerning the cross section of the elements we here will use elements with a constant cross section calculated at the center of each element from the given expression. That is, \( A_1 = A(x = L/6) = 121A_0/144 \), \( A_2 = A(x = L/2) = 81A_0/144 \) and \( A_3 = A(x = 5L/6) = 49A_0/144 \) which gives the following element stiffness matrices
\[
K^e_1 = \frac{121EA_0}{48L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad K^e_2 = \frac{81EA_0}{48L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad K^e_3 = \frac{49EA_0}{48L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.
\]

Before assembling the global stiffness equations \( K\mathbf{a} = \mathbf{f} \) we select a global node numbering in the unknown vector \( \mathbf{a} \) as follows
\[
\mathbf{a} = \begin{cases} a_1 \\ a_2 \\ a_3 \end{cases}
\]
and the Boolean matrices $C_e^i$ can be identified as follows

$$C_e^1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad C_e^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad C_e^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. $$

Rewriting of the global stiffness equation into a sum of expanded element stiffness matrices gives

$$\sum_{i=1}^{3} C_e^T i K_e a = f$$

which ends up in the following system to solve

$$EA_0 \frac{F L}{48L} \begin{bmatrix} 121 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -81 & 81 & 0 \\ 81 & -81 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 49 & -49 \\ 0 & -49 & 49 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F \end{bmatrix}$$

where the solution is

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{160083 \cdot E A_0} \begin{bmatrix} 63504 \\ 158368 \\ 315184 \end{bmatrix}.$$ 

Finally, we can calculate strains and stresses in the elements from

$$\sigma_e^i = E\varepsilon_e^i = E B^e a_e^i = \frac{E}{L_e} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} a_e^1 \\ a_e^2 \end{bmatrix}$$

which gives

$$\sigma_e^1 = 3E \frac{F L}{L \cdot 160083 E A_0} \begin{bmatrix} 0 \\ 63504 \end{bmatrix} = 63504 F/53361 A_0$$

$$\sigma_e^2 = 3E \frac{F L}{L \cdot 160083 E A_0} \begin{bmatrix} 63504 \\ 158368 \end{bmatrix} = 94864 F/53361 A_0$$

$$\sigma_e^3 = 3E \frac{F L}{L \cdot 160083 E A_0} \begin{bmatrix} 158368 \\ 315184 \end{bmatrix} = 158816 F/53361 A_0.$$

Please conclude that both the strain and the stress approximations inside this element type is constant. If we move over to 3-node element one should expect a piece-wise linear variation.

Let us now put in the numerical values $L = 1.0 \ m$, $F = 10000$, $N, E = 2.0 \cdot 10^{11} \ Pa$ and $A_0 = 0.01 \ m^2$ which generates the following numerical results

$$a_1 \simeq 0.1983 \cdot 10^{-5} \ m \quad a_2 \simeq 0.4946 \cdot 10^{-5} \ m \quad a_3 \simeq 0.9844 \cdot 10^{-5} \ m$$
\[ \sigma_1 \simeq 0.1190 \cdot 10^7 \text{ Pa} \quad \sigma_2 \simeq 0.1778 \cdot 10^7 \text{ Pa} \quad \sigma_3 \simeq 0.2939 \cdot 10^7 \text{ Pa} \]

and the analytical results are

\[ u(L/3) = 0.2 \cdot 10^{-5} \text{ m} \quad u(2L/3) = 0.5 \cdot 10^{-5} \text{ m} \quad u(L) = 1.0 \cdot 10^{-5} \text{ m} \]

\[ \sigma(L/6) = 0.1190 \cdot 10^7 \text{ Pa} \quad \sigma(L/2) = 0.1778 \cdot 10^7 \text{ Pa} \quad \sigma(5L/6) = 0.2939 \cdot 10^7 \text{ Pa} \]

The same analysis has been done by the TRINITAS program and results are shown in figure 3 below. All freedom perpendicular to the bar have been fixed because there is no 1D bar element implemented in the program.

![Figure 3: A TRINITAS analysis](image)

**Remarks:**

- The approximation of the displacement always gives the best agreement at the nodes. In this case we have a relative error about one percent.

- A comparison of the stresses at the center of the element show an exact agreement. It can be shown that this is not just luck in this case and that this always holds for exactly this type problem. In the general case one should conclude that the stresses are less accurately approximated compared to the displacements.

- The stress field is in the general case always discontinuous at the element borders and no equilibrium equation in stresses is fulfilled element boundaries.