A 3-Node Element Stiffness Matrix

The selection of shape functions discussed so far is actually the simplest possible with its piece-wise linear nature with a discontinuous first-order derivative. Let us now introduce a second choice of shape functions, still with a discontinuous first-order derivative, requiring a node a the mid-point of each element. By doing so our approximation of the displacement \( u(x) \) will be enhanced by a second-order term and the approximation will be a piece-wise parabolic polynomial chain.

In a typical element of length \( L^e = x_{i+2} - x_i \) we have now defined three element local shape functions in accordance to the figure 1 and element local displacement approximation can be written as

\[
T(\xi) = N_1^e(\xi)a_1^e + N_2^e(\xi)a_2^e + N_3^e(\xi)a_3^e = \mathbf{N}^e \mathbf{a}^e.
\]  

where

\[
\mathbf{N}^e = [ N_1^e(\xi) \quad N_2^e(\xi) \quad N_3^e(\xi) ] = \begin{bmatrix} \xi/2(\xi - 1) & 1 - \xi^2 & \xi/2(\xi + 1) \end{bmatrix}.
\]  

It is now possible to evaluate the \( \mathbf{B}^e \) matrix as follows

\[
\mathbf{B}^e = \left[ \begin{array}{ccc} dN_1^e/ dx & dN_2^e/ dx & dN_3^e/ dx \\ dN_1^e/ d\xi & dN_2^e/ d\xi & dN_3^e/ d\xi \end{array} \right] = \frac{2}{L_e} \left[ \begin{array}{ccc} dN_1^e/d\xi & dN_2^e/d\xi & dN_3^e/d\xi \end{array} \right].
\]  

where the \( \mathbf{B}^e \) matrix in this parabolic case will be dependent from the local coordinate system. After introducing derivatives of the shape functions with respect to \( \xi \) we have

\[
\mathbf{B}^e = \frac{2}{L_e} \left[ \begin{array}{ccc} \xi - 1/2 & -2\xi & \xi + 1/2 \end{array} \right].
\]  

The element stiffness matrix \( \mathbf{K}^e \) will in this case be a 3x3 matrix and in a case with constant cross section and Young’s modulus we receive

\[
\mathbf{K}^e = \frac{4AE}{L_e^3} \int_{-1}^{1} \left[ \begin{array}{ccc} \xi - 1/2 & -2\xi & \xi + 1/2 \\ -2\xi & \xi - 1/2 & -2\xi \\ \xi + 1/2 & -2\xi & \xi - 1/2 \end{array} \right] \frac{L_e}{2} d\xi
\]  

\[18\]
By solving of six different integral over polynomials in $\xi$ we end up in a element stiffness matrix for a 1D 3-node element for second-order problem as it is defined in the box below.

\[
K^e = \frac{2AE}{L^e} \int_{-1}^{1} \begin{bmatrix}
(\xi - 1/2)^2 & -2\xi(\xi - 1/2) & (\xi - 1/2)(\xi + 1/2) \\
4\xi^2 & -2\xi(\xi + 1/2) & (\xi + 1/2)^2 \\
sym.
\end{bmatrix} d\xi \tag{19}
\]

Box: ‘1D 3-node bar element stiffness matrix’

\[
\begin{array}{c|ccc}
 & 14 & -16 & 2 \\
\hline
14 & -16 & 32 & -16 \\
-16 & 32 & -16 & 14
\end{array}
\]