PLASTICITY. Flow rule for isotropic hardening

For a perfectly plastic material we had the following yield hypothesis:

\[ f(\sigma_{ij}) = 0, \]  

which in the von Mises case became

\[ f(\sigma_{ij}) = \sigma_e(\sigma_{ij}) - \sigma_s = \frac{3}{2} s_{ij} s_{ij} - \sigma_s = 0 \]  

For an isotropically hardening material, we will, instead, have

\[ f[\sigma_{ij}, \kappa(\epsilon_{ij}^p)] = 0, \]  

which in the von Mises case can be written

\[ f[\sigma_{ij}, \kappa(\epsilon_{ij}^p)] = \sigma_e(\sigma_{ij}) - \sigma_f(\epsilon_{ij}^p) = \frac{3}{2} s_{ij} s_{ij} - \sigma_f(\epsilon_{ij}^p) = 0, \]  

where \( \sigma_f(\epsilon_{ij}^p) \) is a scalar (called flow stress) that increases monotonically with the plastic deformation. In the von Mises case it is evident that

- \( \sigma_f(\epsilon_{ij}^p) \) is the instantaneous plastic yield limit,
- we still keep the main diagonal of the stress space as a symmetry axis of the yield surface and
- we keep the circular-cylindrical shape of the yield surface.

What happens as the plastic flow develops is that since \( \sigma_f(\epsilon_{ij}^p) \) increases, the diameter of the von Mises cylinder increases. See Fig. 1! This is why this plastic hardening is called isotropic hardening.

![Isotropic hardening (von Mises)](image)

The most frequent isotropic hardening description results from setting

\[ \sigma_f(\epsilon_{ij}^p) = \sigma_f(\epsilon_e^p) \]  

Fig. 1  Isotropic hardening (von Mises)

Mises cylinder increases. See Fig. 1! This is why this plastic hardening is called isotropic hardening.
In case of von Mises,

$$\epsilon^p_e = \int_0^{\epsilon^p_{ij}} d\epsilon^p_e = \int_0^{\epsilon^p_{ij}} \sqrt{\frac{2}{3}} d\epsilon^p_i d\epsilon^p_j$$

(From earlier, we know that the definition of $d\epsilon^p_e$ is such that $dW = \sigma_{ij} d\epsilon^p_{ij} = \sigma_f d\epsilon^p_e$). Note in particular that Eq. (5) shows that one accumulates $\epsilon^p_e$ during the whole history of plastic strains (even during unloading, since $d\epsilon^p_{ij}$ is squared in the definition). By this, we can therefore say that $\sigma_f = \sigma_f[\text{history of } \epsilon^p_{ij}]$.

**Determination of $d\Lambda$ for the von Mises case**

The definition of $d\epsilon^p_e$ [for instance, Eq. (6)] and the general flow rule

$$d\epsilon^p_{ij} = d\Lambda \cdot \frac{\partial f}{\partial \sigma_{ij}}$$

(7)

gives a simple relation between $d\Lambda$ and $d\epsilon^p_e$:

$$d\epsilon^p_e = \sqrt{\frac{2}{3}} d\Lambda \cdot \frac{\partial f}{\partial \sigma_{ij}} d\Lambda \cdot \frac{\partial f}{\partial \sigma_{ij}} = d\Lambda \left[ \frac{2}{3} \cdot \frac{3s_{ij}}{2 \sigma_e} \cdot \frac{3s_{ij}}{2 \sigma_e} \right] = d\Lambda \left[ \frac{3}{2} \sigma^2_{ij} \right] = d\Lambda ,$$

(8)

**Flow rule, isotropic hardening, von Mises**

Combining Eqs. (9), (11) and (13) gives the flow rule

$$d\epsilon^p_{ij} = d\epsilon^p_e \cdot \frac{3s_{ij}}{2 \sigma_e} = d\epsilon^p_e \cdot \frac{3s_{ij}}{2 \sigma_f}$$

(9)

where the last equality comes from the fact that during plastic flow, $\sigma_e = \sigma_f$. Eq. (9) is sometimes called the Prandtl-Reuss equation.

The presence of $d\epsilon^p_e$ in Eq. (9) is awkward, since we would prefer to have the flow rule (which is a constitutive equation) on the form $d\epsilon^p_{ij} = d\epsilon^p_{ij}(d\sigma_{kl})$. If we postulate that

$$\sigma_f = \phi(\epsilon^p_e)$$

(10)

by which

$$d\sigma_f = d\frac{\sigma_f}{d\epsilon^p_e} d\epsilon^p_e = \phi' \cdot d\epsilon^p_e,$$

(11)

Eq. (9) gives

$$d\epsilon^p_{ij} = \frac{d\sigma_f}{\phi'} \cdot \frac{3s_{ij}}{2 \sigma_f}$$

(12)

We therefore want to find $\phi'$. 

Determination of $\phi'$. Uniaxial tensile test

In a standard uniaxial tensile test, during plastic flow we have

$$\sigma_{11} \neq 0; \text{ all other } \sigma_{ij} \equiv 0$$

(13)

and

$$d\epsilon_{11}^p \neq 0; d\epsilon_{22} = d\epsilon_{33} = -\frac{1}{2}d\epsilon_{11}; \text{ all other } d\epsilon_{ij}^p \equiv 0$$

(14)

i.e.,

$$f = \sigma_e - \sigma_f = \sigma_{11} - \sigma_f = 0 \implies \sigma_f = \sigma_{11}$$

(15)

and

$$d\epsilon_e^p = \sqrt{\frac{2}{3}} \left(1 + 2 \cdot (-\frac{1}{2})^2\right) \cdot d\epsilon_{11}^p = d\epsilon_{11}^p$$

(16)

In a uniaxial test, Eq. (11) will therefore simplify into

$$d\sigma_{11} = \phi' \cdot d\epsilon_{11}^p$$

(17)

and the function can be found from the test results.

Suppose, for instance, that the uniaxial test curve of Fig. 2 has been recorded. In the plastic region one then has
\[ d \varepsilon_{11} = d \varepsilon_{11}^e + d \varepsilon_{11}^p = \left( \frac{1}{E} + \frac{1}{\phi'} \right) d \sigma_{11} \Rightarrow \frac{d \sigma_{11}}{d \varepsilon_{11}} = \frac{1}{E + \frac{1}{\phi'}} \Rightarrow \phi' = \frac{\frac{d \sigma_{11}}{d \varepsilon_{11}}}{1 - \frac{1}{E} \frac{d \sigma_{11}}{d \varepsilon_{11}}}. \quad (18) \]

Note, therefore, that the derivative \( d \sigma_{11}/d \varepsilon_{11} \) directly measured in the uniaxial test curve must be 'postprocessed' in order to find \( \phi' \).

In a simple case, this function may be linear, \( i.e., \)

\[ \sigma_f = \phi(\varepsilon_e^p) = \sigma_e + c^{(i)} \varepsilon_e^p \quad (19) \]

\( i.e., \)

\[ \frac{d \sigma_f}{d \varepsilon_e^p} = c^{(i)} \quad (20) \]

The flow rule can then be simplified:

\[ d \varepsilon_{ij}^p = \frac{d \sigma_f}{c^{(i)}} \cdot \frac{3 s_{ij}}{2 \sigma_e} = \frac{d \sigma_e}{c^{(i)}} \cdot \frac{3 s_{ij}}{2 \sigma_e} \quad (21) \]

(the 2\textsuperscript{nd} equality, again, since during plastic flow, \( \sigma_f = \sigma_e \)). Note, again, that a linear stiffness measured in a tensile test must be recalculated in order to find \( c^{(i)} \) (see Eq. (18)!).