PLASTICITY. Flow rule for a stable hardening material

For a perfectly plastic material, the yield condition is

\[ f(\sigma_{ij}) = 0 \]  \hspace{1cm} (1)

For a plastically hardening material, we will, instead, have

\[ f(\sigma_{ij}, \kappa) = 0 \]  \hspace{1cm} (2)

where \( \kappa = \kappa(\epsilon_{ij}^p) \) is a hardening parameter. At this stage, we know nothing about its properties and structure. It can, for instance, be a scalar \( \kappa \) or a tensor \( \kappa_{ij}(\epsilon_{kl}^p) \).

Uniaxial case

In order to establish a theory for a hardening plastic material, again we start by a uniaxial test (Fig. 1)

![Uniaxial tensile test](image)

*Fig. 1* Uniaxial tensile test

We state that the material is stable if

\[ d\sigma \cdot de > 0. \]  \hspace{1cm} (3)

For such a stable material we now study the load cycle of Fig. 2, starting from an elastic stress state \( \sigma^e \), reaching the elastic limit at \( \sigma \), continuing with plastic flow for a load increment \( d\sigma \) and ending with an unloading down to the initial stress state \( \sigma^e \). For this load cycle, \( de = d\epsilon^p \) (see Fig. 2), and the net work input during the cycle is only the plastic work \( W^p \), since the elastic work input during the loading is fully recovered during the unloading. From Fig. 2 we therefore have

\[ W^p = (\sigma - \sigma^e)d\epsilon^p + \frac{1}{2}d\sigma d\epsilon^p > 0 \]  \hspace{1cm} (4)
If we now let \( d\sigma \to 0 \), it is reasonable to assume that also \( d\varepsilon^p \to 0 \), and from Eq. (3) we then get
\[
(\sigma - \sigma^*)d\varepsilon^p \geq 0
\]  
(5)

**Multiaxial case**

Extending the above to the multiaxial case, we study a similar load cycle in a multiaxial case (Fig. 3).

For this, the analogue of Eq. (4) will be
\[
W^p = (\sigma_{ij} - \sigma_{ij}^*)d\varepsilon_{ij}^p + \frac{1}{2}d\sigma_{ij}d\varepsilon_{ij}^p > 0
\]  
(6)

The analogues of Eqs. (3) and (5) are then
\[ d\sigma_{ij}d\epsilon_{ij} > 0 \]  \hspace{1cm} (7)

and

\[ (\sigma_{ij} - \sigma_{ij}^*) d\epsilon_{ij}^P \geq 0 \]  \hspace{1cm} (8)

Eq. (7) is usually called *the principle of maximal plastic work*.

**Flow rule**

Since it must be possible to choose \( \sigma_{ij}^* \) arbitrarily, Eq. (8) must lead to the following two conclusions:

(a) \( d\epsilon_{ij}^P \perp \) the yield surface, and

(b) the yield surface must be convex.

These conclusions can be drawn from the fact that by Eq. (8), all possible \( \sigma_{ij}^* \) must lie behind the border line perpendicular to \( d\epsilon_{ij}^P \) (see Fig. 4). This means that this border line must be tangent to the yield surface. A further consequence is that the yield surface cannot be concave, and since it must be a closed surface enclosing \( \sigma_{ij}^* \), the only possibility left is that it is convex (which in an extreme case can mean that it contains planar surface sections, as, e.g., Tresca’s yield surface). Conclusion (a) above leads directly to the flow rule

\[ d\epsilon_{ij}^P = d\lambda \cdot \frac{\partial f}{\partial \sigma_{ij}} \]  \hspace{1cm} (9)

Note that despite its formal identity with the flow rule of perfect plasticity, Eq. (9) has a somewhat different background. The most important difference is, however, that the multiplier \( d\lambda \) of Eq. (9) is actually a material property, which can be measured, in contrast with the multiplier \( d\lambda \) of perfect plasticity, which remained undetermined and was only given by the boundary conditions of the respective problem.
Consistency condition

During plastic flow, the following consistency conditions must be fulfilled:

\[ \begin{align*}
  f &= 0 \\
  df &= 0
\end{align*} \]  \hspace{1cm} (10)

These will play important roles in the determination of \( d\Lambda \) to follow. \( d\Lambda \)

Sign of \( d\Lambda \)

Eq. (6) must be valid for arbitrary \( \sigma^*_{ij} \); if we therefore choose \( \sigma^*_{ij} = \sigma_{ij} \), we get

\[ d\sigma_{ij} d\varepsilon^p_{ij} > 0, \]  \hspace{1cm} (11)

and Eqs. (9) and (11) give

\[ d\sigma_{ij} \cdot d\Lambda \frac{\partial f}{\partial \sigma_{ij}} > 0. \]  \hspace{1cm} (12)

To see the implications of this, let us study \((\partial f / \partial \sigma_{ij})d\sigma_{ij}\):

1. \((\partial f / \partial \sigma_{ij})d\sigma_{ij} < 0 \quad \Rightarrow \quad \text{unloading (no plastic flow)}\)

2. \((\partial f / \partial \sigma_{ij})d\sigma_{ij} = 0 \quad \Rightarrow \quad \text{neutral loading (no plastic flow)}. \quad \text{Reason: We first assume plastic flow. The second consistency condition then gives} \\
   df = (\partial f / \partial \sigma_{ij})d\sigma_{ij} + (\partial f / \partial \kappa)d\kappa = 0; \quad \text{if we at the same time also have} \\
   (\partial f / \partial \kappa)d\kappa = 0 \quad \Rightarrow \quad d\kappa = 0 \quad \Rightarrow \quad \text{no plastic flow. Our first assumption therefore leads to a contradiction, and the conclusion is no plastic flow.}\n
We have therefore proved that

\[ (\partial f / \partial \sigma_{ij})d\sigma_{ij} > 0 \]  \hspace{1cm} (13)

and by combining Eqs. (12) and (13) we see that

\[ d\Lambda > 0 \]  \hspace{1cm} (14)