PLASTICITY. Return algorithm – linear isotropic von Mises

One problem still remaining unsolved is the problem of *satisfying the yield condition at the end of each iteration*. This requirement is fundamental, since the whole plasticity theory relies on this. This is done by a stress update routine performed in each FE integration (‘Gauss’) point. In the case of linear hardening von Mises plasticity this stress update becomes comparatively simple.

Elastic trial stress

Assume that a $\Delta \epsilon_{ij}$ has been delivered for the Gauss point from the Newton iteration procedure. One then computes a *trial stress* $\sigma_{ij}^t$:

$$\sigma_{ij}^t = \sigma_{ij}^{(1)} + D_{ijkl} \Delta \epsilon_{kl}$$

(1)

in which $D_{ijkl}$ is the generalised linear elasticity stiffness tensor

$$D_{ijkl} = 2G \left[ \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \frac{\nu}{1 - 2\nu} \delta_{ij} \delta_{kl} \right].$$

(2)

where $G$ is the shear modulus:

$$G = \frac{E}{2(1 + \nu)}$$

(3)

If $\sigma_{ij}^t$ is within the yield surface, $\sigma_{ij}^t$ is the correct stress (at least from the point of view of the actual iteration), and no more action is needed. If, on the other hand, $\sigma_{ij}^t$ is outside the yield surface we must return to the yield surface by a *return algorithm*, as will now be described.

Return algorithm

In this case, we have an elastoplastic deformation,

$$\Delta \epsilon_{kl} = \Delta \epsilon_{kl}^e + \Delta \epsilon_{kl}^p$$

(4)

and, consequently, the actual stress at the end of the increment is not the trial stress $\sigma_{ij}^t$ but instead

$$\sigma_{ij}^{(2)} = \sigma_{ij}^{(1)} + D_{ijkl} \Delta \epsilon_{ij}^e = \sigma_{ij}^{(1)} + D_{ijkl} (\Delta \epsilon_{kl} - \Delta \epsilon_{kl}^p) = \sigma_{ij}^t - D_{ijkl} \Delta \epsilon_{kl}^p$$

(5)

where

$$\Delta \epsilon_{kl}^p = \int_c^2 \frac{\partial f}{\partial \sigma_{kl}} dA.$$
whereby Eq. (3) can be formally written

\[ \sigma^{(2)}_{ij} = \sigma^{r}_{ij} - \sigma_{ij}. \]  

Since \( \Delta \varepsilon_{kk}^{p} = 0, \Delta \varepsilon_{kk}^{p} \) is deviatoric, Eq. (7) therefore shows that the return stress \( \sigma^{r}_{ij} \) must be in the deviatoric plane (figure plane of Fig. 1) and

\[ \sigma^{(2)}_{kk} = \sigma^{r}_{kk}. \]  

We must obviously find a way to evaluate the integral of Eq. (6). Adopting an implicit integration scheme, we have

\[ \Delta \varepsilon_{kl}^{p} = \left( \frac{\partial f}{\partial \sigma_{kl}} \right)^{(2)} dA = \frac{3}{2} \frac{s^{(2)}_{ij}}{\sigma^{(2)}_{e}} dA \]  

We can now use Eq. (5) together with Eq. (10) to find \( \sigma^{(2)}_{ij} \):

\[ \sigma^{(2)}_{ij} = \sigma^{r}_{ij} - D_{ijkl} \frac{3}{2} \frac{s^{(2)}_{kl}}{\sigma^{(2)}_{e}} dA = \sigma^{r}_{ij} - 3G \frac{s^{(2)}_{ij}}{\sigma^{(2)}_{e}} dA \]  

(11)

(where

\[ D_{ijkl}s_{kl} = 2G \left[ \frac{1}{2} \left( \delta_{ik} \delta_{jl}s^{(2)}_{kl} + \delta_{il} \delta_{jk}s^{(2)}_{kl} \right) + \frac{\nu}{1+\nu} \delta_{ij} \delta_{kl}s^{(2)}_{kl} \right] \]

\[ = 2G \left[ \frac{1}{2} \left( s^{(2)}_{ij} + s^{(2)}_{ji} + \frac{\nu}{1+\nu} \delta_{ij}s^{(2)}_{kk} \right) \right] = 2Gs^{(2)}_{ij} \]

has been used). From Eq. (11) we derive the stress deviator \( s^{(2)}_{ij} \).
\[ s^{(2)}_{ij} + \frac{1}{3} \delta_{ij} s^{(2)}_{kk} = \sigma_{ij} + 3G \frac{s^{(2)}_{ij}}{\sigma_e^{(2)}} \frac{dA}{\sigma_e^{(2)}} \Rightarrow s^{(2)}_{ij} = \frac{\sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_{kk}^{(2)}}{1 + 3G \frac{dA}{\sigma_e^{(2)}}} = \frac{s^{(2)}_{ij}}{1 + 3G \frac{dA}{\sigma_e^{(2)}}} \]  

(the last equality by Eq. (9)). We can also eliminate \( \sigma_e^{(2)} \), since Eq. (13) gives

\[ \sigma_e^{(2)} = \frac{3}{2} s^{(2)}_{ij} s^{(2)}_{ij} = \frac{3}{2} s^{(2)} \frac{dA}{\sigma_e^{(2)}} = \frac{\sigma_t}{1 + 3G \frac{dA}{\sigma_e^{(2)}}} \]  

\[ \Rightarrow \sigma_e^{(2)} = \sigma_t - 3GdA \]

which can be reinserted into (13) to give

\[ s^{(2)}_{ij} = \cdots = \frac{s^{(2)}_{ij}}{\sigma_e^{(2)}} (\sigma_t - 3GdA) \]  

So far, we have only used the trial stress [Eq. (5)] and the implicit integration assumption [Eq. (10)]. In order to attain an expression for \( s^{(2)}_{ij} \), we must obviously determine \( dA \).

The plastic flow occurring between the states (1) and (2) leads to an increase in flow stress between the two states so that

\[ \sigma_f^{(2)} = \sigma_f^{(1)} + c^{(i)} \Delta \varepsilon_e^p = \sigma_f^{(1)} + c^{(i)}dA \]  

(since \( d\varepsilon_e^p = dA \)). We also know that at state (2), we have plastic flow, i.e., Eq. (14) can be written as

\[ \sigma_f^{(2)} = \sigma_e - 3GdA \]  

Elimination of \( \sigma_f^{(2)} \) between Eqs. (16) and (17) solves \( dA \):

\[ dA = \frac{\sigma_t - \sigma_f^{(1)}}{3G + c^{(i)}} \]  

This can, finally, be inserted into (15) to give a final expression for \( s^{(2)}_{ij} \):

\[ s^{(2)}_{ij} = \frac{s^{(2)}_{ij}}{\sigma_e} (\sigma_e - 3G \cdot \frac{\sigma_t - \sigma_f^{(1)}}{3G + c^{(i)}}) = \cdots = s^{(2)}_{ij} \left( 1 - \frac{1 - \sigma_f^{(1)}}{3G + c^{(i)}} \right), \]  

where \( G \) has again been inserted according to Eq. (3). Eq. (19) is the stress (deviator) state through which the yield surface must now pass.
Note on return algorithms for other plasticity models

Analogous equations can be set up for other plastic flow models. For all such situations where the hardening is linear, a closed-form solution like Eq. (19) can also be established. In case of nonlinear hardening, a simple expression for $d\Lambda$ cannot be found, and an iterative procedure must be set up instead.