PLASTICITY. Flow rule for kinematic hardening

Obviously, for a reversed loading process like the one in the cyclic loading diagram of Fig. 1, the isotropic hardening will lead to a cyclic test behaviour according to the solid line OABCDE of Fig. 2 (in which the length of line segment BC is the same as that of line segment AB). It is, however, a well-established fact that in most materials there is a Bauschinger effect, by which a reversed loading will
rather follow the dashed line OABC’D’E of Fig. 2. This Bauschinger effect can be described by a kinematic hardening in the following way:

$$f = f(\sigma_{ij}, \alpha_{ij}) = \sigma_s (\sigma_{ij} - \alpha_{ij}) - \sigma_s$$

in which \( \alpha_{ij} \) is a 2\(^{nd}\) order tensor:

$$\alpha_{ij} = \alpha_{ij}(\epsilon_{ikl}^P),$$

often called the backstress, and \( \sigma_s \) is the yield strength of the virgin material. Since Eq. (1) states that after plastic flow, \( \sigma_s \) will now be computed using \( \sigma_{ij} - \alpha_{ij} \) instead of \( \sigma_{ij} \) as argument, we will obviously have a translation of the yield surface. See Fig. 3.

![Original yield surface](image1)

![Yield surface after plastic flow](image2)

Fig. 3  Example of kinematic hardening (von Mises case)

What remains is, therefore, to establish the function \( \alpha_{ij} = \alpha_{ij}(\epsilon_{ikl}^P) \). The two most frequent strategies are

\[
\begin{align*}
\text{(Prager)} & \quad d\alpha_{ij} = c^{(k)} d\epsilon_{ij}^P \\
\text{(Ziegler)} & \quad d\alpha_{ij} = d\mu(\sigma_{ij} - \alpha_{ij})
\end{align*}
\]

where \( c^{(k)} \) is a constant that is characteristic for the material (in analogy with \( c^{(i)} \) in the isotropic hardening) and \( d\mu = d\mu(d\epsilon_{ij}^P) \) is a function of the increment of plastic strain which is also characteristic for the material. To illustrate the difference between the Prager and Ziegler models, we can, for instance, look at the Tresca case shown in Fig. 4. (In the von Mises case, it is, on the other hand, easy to realise that the two models are identical.)

Two important properties of the Prager \( d\alpha_{ij} \) may be noticed. Since \( c^{(k)} \) is a constant, Eq. (3) can be directly integrated to give

$$\alpha_{ij} = c^{(k)} \epsilon_{ij}^P,$$

and, further,

$$\alpha_{kk} = c^{(k)} \epsilon_{kk}^P \equiv 0$$
\[ i.e., \text{the Prager } \alpha_{ij} \text{ is deviatoric:} \]
\[ \alpha'_{ij} = \alpha_{ij} \quad (7) \]

**General flow rule for kinematic hardening**

Again, we start by the consistency condition \( df = 0 \)
\[
df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial f}{\partial \alpha_{ij}} d\varepsilon^p_{kl} d\varepsilon^p_{kl} = 0 \quad (8)
\]

From the definition of \( f \) given in Eq. (1), we can differentiate to find \( \frac{\partial f}{\partial \sigma_{ij}} \) and \( \frac{\partial f}{\partial \alpha_{ij}} \):
\[
\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial \sigma_e}{\partial (\sigma_{ij} - \alpha_{ij})} \frac{\partial (\sigma_{ij} - \alpha_{ij})}{\partial \sigma_{ij}} = \frac{\partial \sigma_e}{\partial (\sigma_{ij} - \alpha_{ij})} \quad (9)
\]
\[
\frac{\partial f}{\partial \alpha_{ij}} = \frac{\partial \sigma_e}{\partial (\sigma_{ij} - \alpha_{ij})} \frac{\partial (\sigma_{ij} - \alpha_{ij})}{\partial \alpha_{ij}} = -\frac{\partial \sigma_e}{\partial (\sigma_{ij} - \alpha_{ij})} \quad (10)
\]

Eqs. (8) and (10) together with the fundamental normality rule
\[
d\varepsilon^p_{ij} = d\Lambda \cdot \frac{\partial f}{\partial \sigma_{ij}}, \quad (11)
\]
which is still valid, gives
\[
df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} - \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial \alpha_{ij}}{\partial \varepsilon^p_{kl}} d\varepsilon^p_{kl} = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} - \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial \alpha_{ij}}{\partial \varepsilon^p_{kl}} d\varepsilon^p_{kl} d\Lambda \frac{\partial f}{\partial \sigma_{kl}} = 0
\]
\[
\Rightarrow \quad d\Lambda = \frac{\frac{\partial f}{\partial \sigma_{mn}} d\sigma_{mn}}{\frac{\partial f}{\partial \sigma_{ij}} \frac{\partial \alpha_{ij}}{\partial \varepsilon^p_{kl}} \frac{\partial f}{\partial \sigma_{kl}}}, \quad (12)
\]
and, consequently,
\[ d\varepsilon_{ij} = d\Lambda \cdot \frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f}{\partial \sigma_{mn}} \frac{d\sigma_{mn}}{d\varepsilon_{ij}} \cdot \frac{\partial f}{\partial \sigma_{pq}} \frac{d\sigma_{pq}}{d\varepsilon_{kl}} \frac{\partial f}{\partial \sigma_{kl}} \]  

(13)

**Specialisation to von Mises**

With

\[ f = \sigma_e (\sigma_{ij} - \alpha_{ij}) - \sigma_s = \frac{3}{\sqrt{2}} (s_{ij} - \alpha'_{ij}) (s_{ij} - \alpha'_{ij}) \]  

(14)

we get

\[ \frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f}{\partial s_{kl}} \frac{\partial s_{kl}}{\partial \sigma_{ij}} = \frac{3}{2\sigma_e} (s_{ij} - \alpha'_{ij}) \]  

(15)

This inserted into the general kinematic hardening flow rule [Eq. (13)] gives

\[ d\varepsilon_{ij}^p = \frac{(s_{mn} - \alpha_{mn}) d\sigma_{mn}}{(s_{pq} - \alpha_{pq}) \frac{\partial \alpha_{pq}}{\partial \varepsilon_{kl}} (s_{kl} - \alpha_{kl})} \cdot (s_{ij} - \alpha_{ij}) \]  

(16)

where we have also used the property that \( \alpha_{ij} \) is deviatoric, i.e., \( \alpha'_{ij} = \alpha_{ij} \) (cf. Eqs. (6) and (7)).

**Prager kinematic hardening**

Using the Prager hypothesis, Eq. (16) can be simplified. By Eq. (3) we get

\[ \frac{\partial \alpha_{pq}}{\partial \varepsilon_{kl}} = \delta_{pk} \delta_{ql} c^{(k)} \]  

(17)

This inserted into the flow rule (15) gives

\[ d\varepsilon_{ij}^p = \frac{(s_{mn} - \alpha_{mn}) d\sigma_{mn}}{(s_{pq} - \alpha_{pq}) \delta_{pk} \delta_{ql} c^{(k)} (s_{kl} - \alpha_{kl})} \cdot (s_{ij} - \alpha_{ij}) \]  

\[ = \frac{(s_{mn} - \alpha_{mn}) d\sigma_{mn}}{c^{(k)}(s_{pq} - \alpha_{pq})(s_{pq} - \alpha_{pq})} \cdot (s_{ij} - \alpha_{ij}) \]  

(18)
Uniaxial tensile test. Determination of $c^{(k)}$

In a uniaxial test we have (as before)

$$\sigma_{11} \neq 0; \sigma_{22} = \sigma_{33} = 0 \Rightarrow s_{11} = \frac{2}{3} \sigma_{11}; \ s_{22} = s_{33} = -\frac{1}{3} s_{11},$$  \hspace{1cm} (19)

$$\epsilon_{11}^p \neq 0; \epsilon_{22}^p = \epsilon_{33}^p = -\frac{1}{2} \epsilon_{11}^p,$$  \hspace{1cm} (20)

and consequently in the Prager kinematic hardening case, since $d \alpha_{ij} = c^{(k)} d \epsilon_{ij}^p$,

$$\alpha_{22} = \alpha_{33} = -\frac{1}{2} \alpha_{11}.$$  \hspace{1cm} (21)

The uniaxial tensile test therefore gives

$$d \epsilon_{11} = \frac{(s_{11} - \alpha_{11}) d\sigma_{11}}{c^{(k)} \left[ (s_{11} - \alpha_{11})^2 + 2 \cdot \left(-\frac{1}{2} s_{11} + \frac{1}{2} \alpha_{11}\right)^2 \right]} \cdot (s_{11} - \alpha_{11})$$

$$= \frac{(s_{11} - \alpha_{11})^2}{c^{(k)} \cdot \frac{3}{2} (s_{11} - \alpha_{11})^2} d\sigma_{11} = \frac{d\sigma_{11}}{\frac{3}{2} c^{(k)}}$$  \hspace{1cm} (22)

or

$$d\sigma_{11} = \frac{3}{2} c^{(k)} \cdot d\epsilon_{11}.$$  \hspace{1cm} (23)

I.e., the hardening constant measured in the uniaxial test is $(3/2) c^{(k)}$ (note the difference between this and the isotropic case).