**J INTEGRAL**

**Definition of J integral**

Study the integral

\[
J = \int_{\Gamma} \left( w n_i - T_i \frac{\partial u_i}{\partial x_1} \right) ds
\]

\[
= \int_{\Gamma} \left( w n_i - \sigma_{ij} n_j \frac{\partial u_i}{\partial x_1} \right) ds
\]

in which \( w \) is the energy per unit volume,

\[
w = \int_0^{\epsilon_{ij}} \sigma_{ij} d\epsilon_{ij} \Rightarrow \sigma_{ij} = \frac{\partial w}{\partial \epsilon_{ij}}
\]

and the integration path \( \Gamma \) is a closed path in an arbitrary plane body (i.e., \( \Gamma \) is not the outer contour of the body).

**Proof of path independence of J**

Study the above-defined integral

\[
J = \int_{\Gamma} \left( w n_i - \sigma_{ij} n_j \frac{\partial u_i}{\partial x_1} \right) ds
\]

We now want to prove that \( J = 0 \), if \( w(x_i), u_i(x_j) \) and \( \sigma_{ij}(x_k) \) are differentiable within \( R \) and if \( \sigma_{ij} = \frac{\partial w}{\partial \epsilon_{ij}} \).

We first use the *Einstein summation convention* and the *divergence theorem*:

\[
J = \int_{\Gamma} \left( w \delta_{ij} n_j - \sigma_{ij} n_j \frac{\partial u_i}{\partial x_1} \right) ds = \int_R \frac{\partial}{\partial x_j} (w \delta_{ij} - \sigma_{ij} \frac{\partial u_i}{\partial x_1}) dR
\]

The integrand can be simplified:

\[
\frac{\partial}{\partial x_j} \left( w \delta_{ij} - \sigma_{ij} \frac{\partial u_i}{\partial x_1} \right) = \frac{\partial w}{\partial x_1} - \frac{\partial \sigma_{ij}}{\partial x_1} \frac{\partial u_i}{\partial x_j} - \sigma_{ij} \frac{\partial}{\partial x_1} \left( \frac{\partial u_i}{\partial x_j} \right)
\]

We further realize that

\[
\frac{\partial w}{\partial x_1} = \frac{\partial w}{\partial \epsilon_{ij}} \frac{\partial \epsilon_{ij}}{\partial x_1} = \sigma_{ij} \frac{\partial \epsilon_{ij}}{\partial x_1},
\]

that (by equilibrium)

\[
\frac{\partial \sigma_{ij}}{\partial x_j} = 0,
\]

and that by the definition of strain

\[
\epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).
\]

\( i.e., \)
We have therefore shown that
\[ \frac{\partial}{\partial x_j} \left( w \delta_{ij} - \sigma_{ij} \frac{\partial u_i}{\partial x_j} \right) = \sigma_{ij} \frac{\partial \epsilon_{ij}}{\partial x_j} - \sigma_{ij} \frac{\partial}{\partial x_j} \left( \frac{\partial u_i}{\partial x_j} \right) = \sigma_{ij} \left[ \frac{\partial \epsilon_{ij}}{\partial x_j} - \frac{\partial}{\partial x_j} \left( \frac{\partial u_i}{\partial x_j} \right) \right] \]
\[ = \sigma_{ij} \frac{\partial}{\partial x_j} \left( \epsilon_{ij} - \frac{\partial u_i}{\partial x_j} \right) = \sigma_{ij} \frac{\partial}{\partial x_j} \left[ \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{\partial u_i}{\partial x_j} \right] \]
\[ = \sigma_{ij} \text{ symmetric} \cdot \frac{\partial}{\partial x_j} \left[ -\frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \right] \equiv 0 \]

We have therefore shown that
\[ J_{r_e} = \int_{r_e} \left( wn - \sigma_{ij} n_j \frac{\partial u_i}{\partial x_1} \right) ds = \int_{r_1 - r_2} \left( wn - \sigma_{ij} n_j \frac{\partial u_i}{\partial x_1} \right) ds = 0, \]
(cf. figure below), which, in turn, means that
\[ J_{r_1} = \int_{r_1} \left( wn - \sigma_{ij} n_j \frac{\partial u_i}{\partial x_1} \right) ds = J_{r_2} = \int_{r_2} \left( wn - \sigma_{ij} n_j \frac{\partial u_i}{\partial x_1} \right) ds, \]
and the path-independence of $J_r$ has been proved.

**Relation between $J$ surrounding a crack tip and $G$**

Study a linearly or nonlinearly elastic plane body containing a crack:
We already know that

\[ G = \frac{d}{dA} (W - U^e) \]

This can be rewritten in a convenient way, using the definitions of \( W \) and \( U^e \):

\[ W - U^e = \left( \int_T T_k u_k ds - \int_R \dot{w} dR \right) B \]

where \( T \) is the closed integration loop shown in the figure (following also the crack flanks in order to close the loop) and \( B \) is the thickness of the plane body.

With \( \frac{d}{dt} = \frac{d}{dA} \frac{dA}{dt} \), this gives

\[ \frac{dA}{dt} \frac{d}{dA} (W - U^e) = \frac{dA}{dt} \left[ \left( \int_T T_k \frac{du_k}{dA} ds - \int_R \frac{dw}{dA} dR \right) B \right] \]

and, consequently,

\[ G = \frac{d}{dA} (W - U^e) = \int_T T_k \frac{du_k}{dA} ds - \int_R \frac{dw}{dA} dR \]

We now want to use this for studying the (virtual) growth \( da \) of a crack and introduce a new coordinate system \( x_1, x_2 \) with its origin at the crack tip. For this,

\[ \begin{align*}
  x_1 &= X_1 + a \\
  x_2 &= X_2
\end{align*} \]

and therefore

\[ \frac{du_k}{da} = \frac{d}{da} \left[ u_k[a, X_i(a)] \right] = \frac{\partial u_k}{\partial a} + \frac{\partial u_k}{\partial X_i} \frac{\partial X_i}{\partial a} + \cdots = \frac{\partial u_k}{\partial a} - \frac{\partial u_k}{\partial X_1} \]

and

\[ \frac{dw}{da} = \cdots = \frac{\partial w}{\partial a} - \frac{\partial w}{\partial X_1} \]

We can now rewrite \( \frac{d}{da} (W - U^e) \):

\[ \frac{d}{da} (W - U^e) = \int_T T_k \left( \frac{\partial u_k}{\partial a} - \frac{\partial u_k}{\partial X_1} \right) ds - \int_R \left( \frac{\partial w}{\partial a} - \frac{\partial w}{\partial X_1} \right) dR \]

But

\[ \int_R \frac{\partial w}{\partial a} dR = \int_R \frac{\partial w}{\partial e_{ij}} \frac{\partial e_{ij}}{\partial a} dR = \int_R \sigma_{ij} \frac{\partial e_{ij}}{\partial a} dR = \left[ \text{principle of virtual work} \right] = \int_T T_i \frac{\partial u_i}{\partial a} ds \]

and therefore

\[ \int_T -T_k \frac{\partial u_k}{\partial X_1} ds + \int_R \frac{\partial w}{\partial X_1} dR = \left[ \text{divergence theorem} \right] = \int_R -T_k \frac{\partial u_k}{\partial X_1} + w_n ds \]

\[ = J \]

or

\[ G = J \]

\( J \) can therefore be considered as a theorem for the computation of \( G \) (under the conditions specified).