1. (1 point)
(a) What type of equation is this
\[ \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \]
(b) Write the equation in full (without using tensor notations) for \( i = y \) and \( j = y \). Explain the meaning and the contents of the expression you give.
(c) What is the relationship between \( \varepsilon_{yz} \) and \( \varepsilon_{zy} \)?
(d) What is the relationship between \( \varepsilon_{yz} \) and \( \gamma_{yz} \)?

Solution and answer here:
(a) Equation of deformations (compatibility equation)
(b) With \( i = y \) and \( j = y \) one obtains the normal strain \( \varepsilon_{yy} \)
\[ \varepsilon_{yy} = \frac{1}{2} \left( \frac{\partial u_y}{\partial y} + \frac{\partial u_y}{\partial y} \right) = \frac{\partial u_y}{\partial y} = \frac{\partial v}{\partial y} \]
where \( u_y = v \) is displacement in the \( y \) direction. The displacement is differentiated with respect to \( y \).
(c) \( \varepsilon_{yz} = \varepsilon_{zy} \)
(d) \( \varepsilon_{yz} = \frac{1}{2} \gamma_{yz} \)

2. (1 point)
Using the compliance method, the energy release rate \( G \) can be determined. What is the compliance of a structure?

Solution and answer here:
The compliance \( C \) is defined as
\[ C = \frac{\Delta}{P} \]
where \( \Delta \) is the generalized displacement (displacement, angle, etc) and \( P \) is the load (generalized force, i.e., force, moment, etc). Thus, \( \Delta \) is the deformation associated with the load \( P \). The compliance is the inverse of the stiffness.
3. (1 point)
Define the Tresca effective (or equivalent) stress $\sigma_e^T$ and explain the Tresca yield criterion.

**Solution and answer here:**
According to the Tresca criterion, the effective stress is the largest principal stress minus the smallest principal stress. Thus

$$\sigma_e^T = \sigma_1 - \sigma_3$$

where $\sigma_1 > \sigma_2 > \sigma_3$.

Yielding will occur when

$$\sigma_e^T = \sigma_Y$$

where $\sigma_Y$ is the yield limit of the material.

4. (1 point)
When is the stair-case method used? Explain the method.

**Solution and answer here:**
When the fatigue limit is going to be determined experimentally, it can be done in a way proposed in the staircase method. Experiments are made at predetermined stress levels. If fatigue failure is obtained at one stress level, then the next test specimen is loaded at a lower stress level. On the other hand, if failure is not obtained (after a given number of loading cycles), then the next test specimen is loaded at a higher stress level. After at least 15 to 20 tests, the number of failures and run-outs are counted. The less frequent event (either failures or run-outs) is analyzed, and conclusions can be drawn on the fatigue limit of the material and the scatter (the standard deviation) of the limit.
5.
For a brittle material, the ultimate strength in tension has been determined to $\sigma_{Ut} = 100$ MPa and the ultimate strength in compression has been determined to $\sigma_{Uc} = 200$ MPa.
(a) What ultimate strength in shear $\tau_U$ is expected for this material. Assume that the Mohr failure criterion can be applied using a straight line in the $\sigma\tau$-diagram.
(b) A stress state is given by $\sigma_x = 50$ MPa, $\sigma_y = 0$, and $\tau_{xy}$. Determine the maximum value of $\tau_{xy}$ that can be allowed if failure should be avoided.

**Solution:**
The figure gives for the two Mohr circles given (for tension and compression, respectively)

\[
\frac{50}{s - 50} = \frac{100}{s + 100}
\]
giving $s = 200$ MPa.
(a) The shear load gives the Mohr circle with centre at the origin. This circle, together with the circle due to the tension load, give

\[
\frac{R}{200} = \frac{50}{150}
\]
giving $R = 67$ MPa. Thus the ultimate strength $\tau_U$ in shear is expected to be 67 MPa.
(b) For $\sigma_x = 50$ MPa, $\sigma_y = 0$, and $\tau_{xy}$ unknown the centre of the Mohr circle will be at $\sigma_c = 25$ MPa. To obtain failure, a circle with centre at $\sigma_c = 25$ MPa should have the radius $R_f$, where $R_f$ is obtained from

\[
\frac{R_f}{175} = \frac{50}{150}
\]
giving $R_f = 58$ MPa. The radius of Mohr’s circle is obtained from

\[
R = R_f = \sqrt{\left(\frac{\sigma_c - \sigma_y}{2}\right)^2 + \tau_{xy}^2}
\]
giving $58 = \sqrt{\left(\frac{50}{2}\right)^2 + \tau_{xy}^2}$

from which $\tau_{xy} = 52$ MPa is solved. Thus, to avoid failure the shear stress $\tau_{xy}$ should be less than 52 MPa.
A loading sequence of a structure with an edge crack (see figure) is composed of \( m \) cycles with pulsating stress 0 to 100 MPa to 0 and one single cycle with pulsating stress 0 to 150 MPa to 0, see the lower figure.

Paris’ law for the material reads

\[
\frac{da}{dN} = 5.1 \times 10^{-12} (\Delta K_1)^{4.17} \text{ m per cycle}
\]

where \( \Delta K_1 \) has the unit MPa m\(^{1/2} \) (and \( n = 4.17 \)).

(a) Which is the minimum crack length \( a_{\text{min}} \) required to give crack propagation at the stress cycle to 100 MPa? The material has threshold value \( \Delta K_{\text{th}} = 8 \) MPa m\(^{1/2} \)

(b) Assuming \( a > a_{\text{min}} \), how many cycles \( m \) to stress level 100 MPa are required to give the same crack growth as one single cycle to stress level 150 MPa?

**Solution:**

(a) The stress intensity range is

\[
\Delta K_1 = \Delta \sigma \sqrt{\pi a} f_3 \left( \frac{a}{W} \right) = \Delta \sigma \sqrt{\pi a} \ 1.12
\]

For crack propagation to occur, the stress intensity range \( \Delta K_1 \) should be larger than the threshold value \( \Delta K_{\text{th}} \). Stress range 100 MPa then gives

\[
\Delta K_1 = 100 \sqrt{\pi a} \ 1.12 = 8 \ ( = \Delta K_{\text{th}})
\]

which gives \( a = 0.0016 \) m (thus, \( a \) should be larger than 1.6 mm)

(b) Paris’ law gives, for a sequence of \( m \) cycles to stress level 100 MPa, the crack growth per sequence

\[
\frac{da}{dN_s} = 5.1 \times 10^{-12} \left\{ m \cdot (100 \sqrt{\pi a} \ 1.12) \right\}^{4.17} \text{ m per sequence}
\]

Paris’ law gives, for one cycle to stress 150 MPa, the crack growth per cycle

\[
\frac{da}{dN} = 5.1 \times 10^{-12} (150 \sqrt{\pi a} \ 1.12)^{4.17} \text{ m per cycle}
\]

The same crack growth in the two cases gives

\[
\frac{da}{dN_s} = 5.1 \times 10^{-12} m \cdot (100 \sqrt{\pi a} \ 1.12)^{4.17} = \left( \frac{da}{dN} \right) 5.1 \times 10^{-12} (150 \sqrt{\pi a} \ 1.12)^{4.17}
\]

which gives \( m = 150^{4.17/100^{4.17}} = 5.4 \).

Thus, 5.4 cycles to stress level 100 MPa give the same crack growth as one cycle to stress level 150 MPa.
The structure in the figure is loaded by a force \( P = P_0 \pm P_0 \). Determine, with respect to fatigue failure at hole A, the maximum allowable value of \( P_0 \). The hole is machined (\( R_a = 7 \) µm) and it has a diameter of 10 mm.

Numerical data: ultimate strength \( \sigma_u = 640 \) MPa, yield limit \( \sigma_Y = 360 \) MPa, fatigue limit \( \sigma_{FL} = 240 \) MPa and \( \sigma_{FLP} \) is unknown (use thumb rule \( \sigma_{FLP} \cong 0.85\sigma_{FL} \)). \( H = 200 \) mm and \( h = 10 \) mm.

**Solution:**

The nominal stress at the hole becomes \( \sigma_{nom} = Mz/I \) where \( M = P \cdot 40H \), \( z = H/2 \) and 

\[
I = \frac{(H + h)^4 - (H - h)^4}{12}.
\]

One obtains \( \sigma_{nom} = 14963P = 14963(P_0 \pm P_0) \) N/m\(^2\) (\( P \) in N).

The stress concentration factor \( K_t \) at the hole is \( K_t = 3.0 \) (we use here the case "a small hole in a large plate"). This gives the fatigue notch factor

\[
K_t = 1 + q(K_t - 1) = 1 + 0.84 \cdot (3 - 1) = 2.68.
\]

The factors reducing the fatigue limit due to surface finish and volume become (approximately): \( \kappa \cong 0.86 \), \( \lambda \cong 1.0 \) and \( \delta = 1 \).

The upper plate (in which the hole is) of the beam cross section is loaded mainly in tension (the bending may be neglected). Use fatigue limit \( \sigma_{FL} \) (rather than \( \sigma_{FLB} \)). Pulsating load implies that the fatigue limit \( \sigma_{FLP} \) should be used. One obtains \( \sigma_{FLP} = 0.85 \sigma_{FL} = 204 \) MPa.

By use of the reduction factors one obtains

\( \sigma_{FLP}^{red} = \kappa\lambda\delta\sigma_{FL}^{red} = \kappa\lambda\delta\cdot 0.85 \sigma_{FL} = 0.86 \cdot 1.0 \cdot 1.0 \cdot 204 \) MPa = 175 MPa,

and \( \sigma_{FL}^{red} = 0.86 \cdot 1.0 \cdot 1.0 \cdot 240 \) MPa = 206 MPa.

The amplitude \( P_0 \) of the load gives the stress amplitude \( K_t\sigma_{nom} = 2.68\cdot\sigma_{nom} \).

Let \( K_t\sigma_{nom} = \sigma_{FLP}^{red} \). It gives

\[2.68\cdot14963P_0 = 175\cdot10^6, \text{ which gives } P_0 = 4.36 \text{ kN}.
\]

**Answer:** If no safety factor is used, the load \( P \) may be \( P = 4.36 \pm 4.36 \) kN.

**Comment:**

When \( K_t\sigma_{nom} = \sigma_{FLP}^{red} \) was used, a small approximation was made. As seen in the Haigh diagram, the service stress will not meet the curve giving the fatigue limit exactly at \( \sigma_{FLP}^{red} = 175 \) MPa. An exact value of the allowed stress amplitude is obtained from the intersection of the two curves

\[ \sigma_a = \frac{2.68}{3.0} \sigma_m \text{ and } \sigma_a = 206 - \frac{31}{204} \sigma_m \]

giving \( K_t\sigma_{nom} = 176 \) MPa, from which almost the same \( P_0 \) as above is obtained.
A flat plate has a *small elliptical* hole, see figure. The major axis $2b$ is perpendicular to the direction of the stress. The minor axis is $2a$, and $b = 2a$. The plate is subjected to the stress $\sigma_{\infty} = 100 \pm 150$ MPa. Estimate the number of stress cycles to fatigue failure by use of Neuber’s method. Assume that $K_f = 0.9K_t$.

Use the Ramberg-Osgood’s material relation for amplitudes ($E = 200$ GPa, $K' = 1344$ MPa, $n' = 0.18$)

$$\varepsilon_a = \frac{\sigma_a}{E} + \left( \frac{\sigma_a}{K'} \right)^{1/n'}$$

with appropriate modifications of the formula for changes $\Delta\sigma$ of the stress.

For the fatigue life analysis, use the Coffin-Manson relation, with numerical values $\sigma_U = 800$ MPa, and $\Psi = 0.65$.

**Solution:**

The stress concentration factor $K_t$ is $(1 + 2b/a) = 5$. No information for calculation of the notch sensitivity factor $q$ is given here. Instead, $K_f = 0.9K_t$ was given. Thus, use $K_f = 0.9 \cdot 5 = 4.5$.

The maximum local stress $\sigma_{\text{max}}$ and the maximum local strain $\varepsilon_{\text{max}}$ at the notch are not needed here, because in Coffin-Manson’s relation the stress mean value is not included. Only the change of strain, i.e. the strain range $\Delta\varepsilon$, is required. This strain range is obtained as the point of intersection between the Neuber hyperbola and the material stress-strain relation for load changes. One obtains, for a nominal change of stress $\Delta\sigma_{\infty} = 300$ MPa (change of stress far away from the stress concentration),

$$\begin{align*}
\Delta\sigma \cdot \Delta\varepsilon &= \frac{K_f^2 (\Delta\sigma_{\infty})^2}{E} = \frac{4.5^2 \cdot 300^2}{200 000} = 9.1125 \\
\Delta\varepsilon &= \frac{\Delta\sigma}{E} + 2 \left( \frac{\Delta\sigma}{2K'} \right)^{1/n'} = \frac{\Delta\sigma}{200 000} + 2 \left( \frac{\Delta\sigma}{2 \cdot 1344} \right)^{1/0.18}
\end{align*}$$

where $\Delta\sigma$ and $\Delta\varepsilon$ are changes of stress and strain at the stress concentration. From this system of equations $\Delta\varepsilon = 0.0098759$ and $\Delta\sigma = 922.7$ MPa are solved.

Now Coffin-Manson’s rule gives

$$\Delta\varepsilon = 3.5 \frac{\sigma_U}{E} (N)^{-0.12} + D^{0.6} (N)^{-0.6} = 3.5 \frac{800}{200 000} (N)^{-0.12} + 1.0498^{0.6} (N)^{-0.6}$$

For $\Delta\varepsilon = 0.0098759$ one obtains $N = 7080$ cycles.

Thus, failure is expected after, approximately, 7 000 cycles.